Convex sets

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Abstract

In these notes I review some facts about convex sets. First I treat when convex combinations of elements are unique. Then I review the Caratheodory theorem. Next I consider extreme elements and minimal sets that generate the convex set. I finish with convex isomorphisms. I assume that the concepts of convex sets and convex hulls are familiar.

Contents

1 Unique convex combinations

Definition 1. Let C be the convex hull of $v_1, \dots, v_m$. An element $v \in C$ has a unique convex combination of elements $v_1, \dots, v_m$ when $\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_k \geq 0, \sum_{i=1}^k \mu_i = \sum_{i=1}^k \lambda_i = 1$ and $v = \lambda_1 v_1 + \ldots + \lambda_m v_m = \mu_1 v_1 + \ldots + \mu_m v_m$ impies $\lambda_i = \mu_i$, for all $i = 1, \dots, m$.

Lemma 2. *Let V be a real vector space. Let $k \in \mathbb{N}$. Let $v_1, \dots, v_{k+1} \in V$ and let C be the convex hull of $v_1, \dots, v_{k+1}$. Then each element of C has a unique convex combination of elements of $v_1, \dots, v_{k+1}$ if and only if $v_1 - v_{k+1}, \dots, v_k - v_{k+1}$ are linearly independent.*

Proof. First we prove that when $v_1 - v_{k+1}, \dots, v_k - v_{k+1}$ are linearly independent, that each element of C has a unique convex combination of elements $v_1, \dots, v_{k+1}$.

Let $v \in C$ and let $v = \lambda_1 v_1 + \ldots + \lambda_{k+1} v_{k+1} = \mu_1 v_1 + \ldots + \mu_{k+1} v_{k+1}$ be convex combinations of v. Then $v - v_{k+1} = \lambda_1(v_1 - v_{k+1}) + \ldots + \lambda_{k+1}(v_{k+1} - v_{k+1}) = \mu_1(v_1 - v_{k+1}) + \ldots + \mu_{k+1}(v_{k+1} - v_{k+1})$, so

$$\lambda_1(v_1 - v_{k+1}) + \ldots + \lambda_k(v_k - v_{k+1}) = \mu_1(v_1 - v_{k+1}) + \ldots + \mu_k(v_k - v_{k+1}).$$

It follows from the fact that $v_1 - v_{k+1}, \dots, v_k - v_{k+1}$ are linearly independent, that $\lambda_i = \mu_i$ for all $i = 1, \dots, k$. Finally, $\lambda_{k+1} = 1 - \lambda_1 - \ldots - \lambda_k = 1 - \mu_1 - \ldots - \mu_k = \mu_{k+1}$. So v has a unique convex combination.

For the proof in the other direction, suppose $v_1 - v_{k+1}, \ldots, v_k - v_{k+1}$ are not linearly independent. We will show, that there is an element in the convex hull that does not have a unique convex combination.

From the linear dependence of $v_1 - v_{k+1}, \ldots, v_k - v_{k+1}$ follows that there are $\alpha_1, \ldots, \alpha_k$, not all zero, so that $\alpha_1(v_1 - v_{k+1}) + \ldots + \alpha_k(v_k - v_{k+1}) = 0$. Let $I = \{i : \alpha_i > 0\}$ and $J = \{i : \alpha_i \leq 0\}$. So

$$\sum_{i \in I} \alpha_i(v_i - v_{k+1}) = \sum_{i \in J} -\alpha_i(v_i - v_{k+1}).$$

As at least one $\alpha_i \neq 0$, $i \in \{1, \ldots, k\}$, at least one of $\sum_{i \in I} \alpha_i$ or $\sum_{i \in J} -\alpha_i$ is positive, and both are non-negative. Let $M = \max\left\{\sum_{i \in I} \alpha_i, \sum_{i \in J} -\alpha_i\right\} > 0$. Let $\beta = M - \sum_{i \in I} \alpha_i$ and $\gamma = M - \sum_{i \in J} -\alpha_i$. Note that $\beta, \gamma \geq 0$, and that $\beta + \sum_{i \in I} \alpha_i = \gamma + \sum_{i \in J} -\alpha_i = M$. As $v_{k+1} - v_{k+1} = 0$, we have

$$\frac{\beta}{M}(v_{k+1} - v_{k+1}) + \sum_{i \in I} \frac{\alpha_i}{M}(v_i - v_{k+1}) = \frac{\gamma}{M}(v_{k+1} - v_{k+1}) + \sum_{i \in J} \frac{-\alpha_i}{M}(v_i - v_{k+1}).$$

Using that $\frac{\beta}{M} + \sum_{i \in I} \frac{\alpha_i}{M} = \frac{\gamma}{M} + \sum_{i \in J} \frac{-\alpha_i}{M} = 1$, adding v_{k+1} on both sides gives

$$\frac{\beta}{M} v_{k+1} + \sum_{i \in I} \frac{\alpha_i}{M} v_i = \frac{\gamma}{M} v_{k+1} + \sum_{i \in J} \frac{-\alpha_i}{M} v_i.$$

As I and J are disjoint, and at least one of $\alpha_i \neq 0$, it follows that this are two different convex combinations of $v_1, \ldots, v_{k+1}$ of the same element $\frac{\beta}{M} v_{k+1} + \sum_{i \in I} \frac{\alpha_i}{M} v_i$. $\square$

Lemma 3. *Let V be a vector space, and C the convex hull of $v_1, \ldots, v_m \in V$. When $v \in C$ has two different convex combinations of $v_1, \ldots, v_m$, then v has infinitely many convex combinations of $v_1, \ldots, v_m$.*

Proof. Suppose

$$v = \sum_{i=1}^{m} \lambda_i v_i = \sum_{i=1}^{m} \mu_i v_i$$

are two different convex combinations of v. So for some $i_0 \in \{1, \ldots, m\}$, $\lambda_{i_0} \neq \mu_{i_0}$. Let $\alpha \in [0, 1]$. Note that

$$v = \sum_{i=1}^{m} (\alpha \lambda_i + (1 - \alpha)\mu_i) v_i =: \sum_{i=1}^{m} \nu_i(\alpha) v_i,$$

is also a convex combination of v. When $\alpha_1 \neq \alpha_2$, $\nu_{i_0}(\alpha_1) - \nu_{i_0}(\alpha_2) = (\alpha_1 - \alpha_2)(\lambda_{i_0} - \mu_{i_0}) \neq 0$. Hence there are infinitely many convex combinations of v. $\square$

Definition 4. Let C be a convex set. A convex combination

$$v = \sum_{i=1}^{m} \lambda_i v_i$$

is open when for all $i \in \{1, \ldots, m\}$, $\lambda_i > 0$.

Definition 5. Let V be a real vector space and let $v_1, \ldots, v_m \in V$. We define the open convex set generated by $v_1, \ldots, v_m$ to be the set of all open convex combinations of $v_1, \ldots, v_m$.

Lemma 6. *Let V be a real vector space and let C° be the open convex set generated by $v_1, \ldots, v_m \in V$. Let C be the convex set generated by $v_1, \ldots, v_m$. Then C° is convex and $\emptyset \neq C^\circ \subseteq C$.*

Proof. It is obvious that C° is contained in the convex set generated by $v_1, \ldots, v_m$. We have that $(1/m)v_1 + \ldots + (1/m)v_m \in C^\circ$, so C° is not empty.

Let $v = \sum_{i=1}^m \lambda_i v_i, w = \sum_{i=1}^m \mu_i v_i \in C^\circ$, $\lambda_i, \mu_i > 0$ for all i. Let $\alpha \in [0,1]$. Then

$$\alpha v + (1-\alpha)w = \sum_{i=1}^m (\alpha\lambda_i + (1-\alpha)\mu_i)v_i.$$

Note that $\sum_{i=1}^m (\alpha\lambda_i + (1-\alpha)\mu_i) = 1$, and $\alpha\lambda_i + (1-\alpha)\mu_i > 0$, for all $i \in \{1, \ldots, m\}$. Hence $\alpha v + (1-\alpha)w \in C^\circ$. So C° is convex. □

Lemma 7. *Let V be a vector space and let C° (resp. C) be the open (resp. closed) convex set generated by $v_1, \ldots, v_m \in V$. The following statements are equivalent:*

(i) There is an element $v \in C$ that does not have a unique convex combination of $v_1, \ldots, v_m$.

(ii) Every element of $v \in C^\circ$ does not have a unique convex combination of $v_1, \ldots, v_m$.

(iii) For every element $v \in C^\circ$ there are infinitely many convex combinations of $v_1, \ldots, v_m$.

Proof. Obviously, (iii) $\implies$ (ii). As C° is not empty (lemma 6), (ii) $\implies$ (i). The implication (ii) $\implies$ (iii) follows from lemma 3. We are only left to prove (i) $\implies$ (ii). Let $v \in C$ be an element so that

$$v = \sum_{i=1}^m \lambda_i v_i = \sum_{i=1}^m \mu_i v_i$$

are two different convex combinations of v. Note that

$$0 = \sum_{i=1}^m (\lambda_i - \mu_i)v_i.$$

Let $w \in C^\circ$ have an open convex combination

$$w = \sum_{i=1}^m \nu_i v_i.$$

Let $\alpha = \min_i \nu_i > 0$. As $\lambda_i - \mu_i \geq -1$, $\nu_i + \alpha(\lambda_i - \mu_i) \geq 0$, for all i, and $\sum_{i=1}^m (\nu_i + \alpha(\lambda_i - \mu_i)) = \sum_{i=1}^m \nu_i + \alpha \sum_{i=1}^m (\lambda_i - \mu_i) = 1 + 0 = 1$. So

$$w = \sum_{i=1}^m (\nu_i + \alpha(\lambda_i - \mu_i))v_i.$$

is another convex combination of w, because for at least one $i \in \{1, \ldots, m\}, \lambda_i \neq \mu_i$. □

2 Theorem of Carathéodory (1907)

Theorem 8 (Carathéodory)**.** *Let V be a real n-dimensional vector space. Let C be the convex hull of a set S. Then each element x is the convex combination of at most $n+1$ elements in S.*

Proof. Let $y \in C$. Let m be the smallest integer so that there are $x_1, \ldots, x_m \in C$ and $\lambda_1, \ldots, \lambda_m > 0$, $\sum_{i=1}^m \lambda_i = 1$ so that

$$y = \sum_{i=1}^m \lambda_i x_i.$$

Suppose $m > n+1$. As V is n-dimensional, there are scalars $\alpha_2, \ldots, \alpha_m$, at least one of them positive, so that

$$0 = \sum_{i=2}^{m} \alpha_i (x_i - x_1).$$

Let $\alpha_1 = -\sum_{i=2}^{m} \alpha_i$. Then $\sum_{i=1}^{m} \alpha_i = 0$. It follows that

$$0 = \sum_{i=2}^{m} \alpha_i (x_i - x_1) = \alpha_1 x_1 + \sum_{i=2}^{m} \alpha_i x_i = \sum_{i=1}^{m} \alpha_i x_i.$$

Let $\mu = \min_{i:\alpha_i>0} \frac{\lambda_i}{\alpha_i}$, and let $j \in \{1, \ldots, m\}$ be such that $\mu = \frac{\lambda_j}{\alpha_j}$ and $\alpha_j > 0$. For all $i \in \{1, \ldots, m\}$, $\lambda_i - \mu\alpha_i \geq 0$, and $\lambda_j - \mu\alpha_j = 0$. Moreover, we have

$$\sum_{i \neq j} (\lambda_i - \mu\alpha_i) = \sum_{i=1}^{m} (\lambda_i - \mu\alpha_i) = \sum_{i=1}^{m} \lambda_i - \mu \sum_{i=1}^{m} \alpha_i = 1 - 0 = 1.$$

It follows that

$$\sum_{i \neq j} (\lambda_i - \mu\alpha_i) x_i = \sum_{i=1}^{m} (\lambda_i - \mu\alpha_i) x_i = \sum_{i=1}^{m} \lambda_i x_i - \mu \sum_{i=1}^{m} \alpha_i x_i = y - 0 = y$$

is a convex combination of y with less than m elements, which is in contradiction with our assumption that m was the minimum number of elements of S needed to represent y as a convex combination of elements in S. It follows that each element in C can be represented with at most $n+1$ elements from S. $\square$

3 Minimal sets and extreme elements

Definition 9. Let S be a subset of a vector space, and let C be the convex space generated by S. We call S minimal, when for every $x \in S$, $C \neq \mathrm{co}(S \setminus \{x\})$.

Not every convex set has a minimal generating set.

Example 10. *Consider the real numbers $\mathbb{R}$, and let S generated $\mathbb{R}$. First note that S is infinite, as otherwise $r = \max_{x \in S} |x| < \infty$ and $\mathrm{co}(S) \subseteq [-r, r] \neq \mathbb{R}$. Let $x, y, z \in S$ so that $x < y < z$. Note that y is a convex combination of x and z, so $S \setminus \{y\}$ also generates $\mathbb{R}$. So S is not minimal.*

Definition 11. Let C be convex and $x \in C$. We call x extreme, when there are no $y, z \in C$, $y \neq z$ and $\alpha \in (0,1)$ so that $x = \alpha y + (1-\alpha) z$.

Lemma 12. *Let C be a convex set generated by a minimal set S. Then S is the set of all extrema of C.*

Proof. For an extremum $x \in C$, there are no $y, z \in C, y \neq z$ and $\alpha \in (0,1)$ so that $x = \alpha y + (1-\alpha) z$. So $\mathrm{co}(S \setminus \{x\})$ does not contain x. Hence $x \in S$.

Suppose $x \in S$ is not extreme. Then there are $y, z \in C, x \neq y$ and $\alpha \in (0,1)$ so that $x = \alpha y + (1-\alpha) z$. Then there are elements $x_1, \ldots, x_m \in S$ so that y, z are convex combinations

$$y = \sum_{i=1}^{m} \beta_i x_i, \quad \text{and} \quad z = \sum_{i=1}^{m} \gamma_i x_i.$$

Note that we can choose this x_i so that at least one of β_i or γ_i is positive. So

$$x = \sum_{i=1}^{m} (\alpha\beta_i + (1-\alpha)\gamma_i) x_i.$$

If all $x_i \neq x$, then x is a convex combination of other elements of S, and so $C = \mathrm{co}(S \setminus \{x\})$, so S is not minimal. Contradiction. So x is equal to some x_i. After relabelling, if necessary, we may assume $x = x_1$. As $y \neq z$, either $\beta_1 < 1$ or $\gamma_1 < 1$, or both. We already saw that $\beta_1 > 0$ or $\gamma_1 > 0$. So $0 < \alpha\beta_1 + (1-\alpha)\gamma_1 < 1$. So

$$(1 - (\alpha\beta_1 + (1-\alpha)\gamma_1))x = \sum_{i=2}^{m} (\alpha\beta_i + (1-\alpha)\gamma_i)x_i.$$

Note that $\sum_{i=2}^{m} (\alpha\beta_i + (1-\alpha)\gamma_i) = 1 - (\alpha\beta_1 + (1-\alpha)\gamma_1)$, so

$$x = \sum_{i=2}^{m} \frac{\alpha\beta_i + (1-\alpha)\gamma_i}{1 - (\alpha\beta_1 + (1-\alpha)\gamma_1)} x_i,$$

is a convex combination of elements from $S \setminus x$. So $C = \mathrm{co}(S \setminus \{x\})$, so S is not minimal. Contradiction. It follows that all elements of S are extreme. $\square$

Vice versa, a set of extreme points does not necessarily generate the convex set.

Example 13. *Consider*

$$C = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, \ \textit{when } x, y \geq 0, \ \textit{otherwise } x^2 + y^2 < 1\right\}.$$

Then C is convex, and the set of extreme points is

$$E = \left\{(x, y) : x^2 + y^2 = 1, x \geq 0, y \geq 0\right\}.$$

However C is not generated by E.

A corollary to lemma 12 is

Corollary 14. *A convex set C has at most one minimal set. When it exist, it is unique, which allows us to speak about the minimal set.*

Proof. If C has a minimal set, then it is the set of extrema, which uniquely determines the minimal set. $\square$

Lemma 15. *When C is a convex set generated by its set E of extrema, then E is the minimal set.*

Proof. Suppose E is not minimal, then there is an $x \in E$ so that C is generated by $E \setminus \{x\}$. So there are $x_1, \ldots, x_m \in E \setminus \{x\}$ and $\lambda_1, \ldots, \lambda_m > 0$, $\sum_{i=1}^{m} \lambda_i = 1$ so that $x = \sum_{i=1}^{m} \lambda_i x_i$. But then x is not extreme. Contradiction. $\square$

A corollary to lemmas 12 and 15 is

Corollary 16. *Let C be a convex set generated by $S \subseteq C$. Then S is minimal if and only if S is the set of all extreme points.*

However not every element in a convex set C generated by a minimum set S has necessarily a unique decomposition of elements in S.

Example 17. *Take for instance $C = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\}$, which has minimal set $S = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\}$. Then*

$$(0, 0) = \frac{1}{2}(-1, 0) + \frac{1}{2}(1, 0) \quad \textit{and} \quad (0, 0) = \frac{1}{2}(0, -1) + \frac{1}{2}(0, 1).$$

However, if C is generated by $S = \{x_1, \ldots, x_m\}$ and every element in C has a unique decomposition in terms of elements of S, then S is minimal:

Lemma 18. *Let C be a convex set generated by $S = \{x_1, \ldots, x_m\}$. If every element in C has a unique decomposition in terms of S, then S is a minimal set.*

Proof. Let $x_j \in S$. Suppose there are $y, z \in C$ and $\alpha \in (0, 1)$ so that

$$x_j = \alpha y + (1 - \alpha)z.$$

Then y and z have convex decompositions

$$y = \sum_{i=1}^{m} \beta_i x_i, \quad \text{and} \quad z = \sum_{i=1}^{m} \gamma_i x_i.$$

So

$$x_j = \sum_{i=1}^{m} (\alpha\beta_i + (1 - \alpha)\gamma_i)x_i,$$

is a convex decomposition of x_j in terms of $x_1, \ldots, x_m$. As the convex decompositions are unique, $\beta_i = \gamma_i = 0$ for all $i \neq j$, and $\beta_j = \gamma_j = 1$, so $y = z$, so x_j is extreme. So by corollary 16 S is a minimal set. □

Lemma 19. *When C is generated by a finite set, then C has a minimal set.*

Proof. Let S be a set of minimum cardinality that generates C, then S is minimal, because for $x \in S$, $S \setminus \{x\}$ has a lower cardinality than S and therefore does not generate C. □

Corollary 20. *When C does not have extreme points, then C is not generated by a finite set.*

4 Convex isomorphisms

Definition 21. Let C_1, C_2 be convex sets. We call a map $f : C_1 \to C_2$ a convex homomorphism when $f(\alpha x + (1 - \alpha)y) = \alpha f(x) + (1 - \alpha)f(y)$ for all $\alpha \in [0, 1]$.

Lemma 22. *When $f : C_1 \to C_2$ is a bijective convex homomorphism, f^{-1} is also a convex homomorphism.*

Proof. Let $x, y \in C_2$ and $\alpha \in [0, 1]$. There are $a, b \in C_1$ so that $x = f(a), y = f(b)$. So

$$\begin{aligned} f^{-1}(\alpha x + (1 - \alpha)y) =& f^{-1}(\alpha f(a) + (1 - \alpha)f(b)) \\ =& f^{-1}(f(\alpha a + (1 - \alpha)b)) \\ =& \alpha a + (1 - \alpha)b \\ =& \alpha f^{-1}(f(a)) + (1 - \alpha)f^{-1}(f(b)) \\ =& \alpha f^{-1}(x) + (1 - \alpha)f^{-1}(y). \end{aligned}$$

So f^{-1} is convex. □

Definition 23. A convex isomorphism is a bijective convex homomorphism.

Remark 24. Note that a convex isomorphism $f : C_1 \to C_2$ maps extreme points to extreme points. When C_1 is generated by S_1, then C_2 is generated by $f(S_1)$. When S_1 is minimal, then $f(S_1)$ is minimal. When every element in C_1 has a unique convex decomposition in terms of S_1, then every element of C_2 has a unique decomposition in terms of S_2.

Definition 25. Let C_i be a convex subset of a normed vector space V_i, $i = 1, 2$. A convex isometrism is a convex isomorphism $f : C_1 \to C_2$ so that for all $x, y \in C_1$, $\|x - y\| = \|f(x) - f(y)\|$.

Lemma 26. *Let V, W be linear spaces and $x_1, \dots, x_{k+1} \in V, y_1, \dots, y_{k+1} \in W$, so that $x_1 - x_{k+1}, \dots, x_k - x_{k+1}$ are linearly independent in V and $y_1 - y_{k+1}, \dots, y_k - y_{k+1}$ are linearly independent in W. Let C be the convex hull of $x_1, \dots, x_{k+1}$ and D the convex hull of $y_1, \dots, y_{k+1}$. Then $f : C \to D$ defined by*

$$f\left(\sum_{i=1}^{k+1} \alpha_i x_i\right) = \sum_{i=1}^{k+1} \alpha_i y_i$$

is a well-definined convex isomorphism.

When V and W are inner product spaces and $x_1 - x_{k+1}, \dots, x_k - x_{k+1}$ are orthogonal in V and $y_1 - y_{k+1}, \dots, y_k - y_{k+1}$ are orthogonal in W, and $0 < \|x_i - x_{k+1}\| = \|y_i - y_{k+1}\|$, for all i, then f is a well-defined convex isometrism.

Proof. By lemma 2 any element in C has a unique convex decomposition in terms of $x_1, \dots, x_{k+1}$, so f is well defined. By construction it is also a convex homomorphism and it is surjective. Now suppose $f(a) = f(b)$. There are unique convex decompositions

$$a = \sum_{i=1}^{k+1} \alpha_i x_i \quad \text{and} \quad b = \sum_{i=1}^{k+1} \beta_i x_i. \tag{1}$$

Using that f is a convex homomorphism, and that $f(x_i) = y_i$ for all i,

$$f(a) = \sum_{i=1}^{k+1} \alpha_i y_i = \sum_{i=1}^{k+1} \beta_i y_i = f(b).$$

As all elements in D have a unique convex decomposition, $\alpha_i = \beta_i$, for all i. It follows that $a = b$. So f is injective as well. So f is a convex isomorphism.

Now suppose that V and W are inner product spaces and $x_1 - x_{k+1}, \dots, x_k - x_{k+1}$ are orthogonal in V and $y_1 - y_{k+1}, \dots, y_k - y_{k+1}$ are orthogonal in W, and $0 < \|x_i - x_{k+1}\| = \|y_i - y_{k+1}\|$, for all i. In particular $x_1 - x_{k+1}, \dots, x_k - x_{k+1}$ are linearly independent in V, and $y_1 - y_{k+1}, \dots, y_k - y_{k+1}$ are linearly independent in W. So $f : V \to W$ is a convex isomorphism. Let $a, b \in C$, with unique convex decompositions as in eq. (1). Then

$$\begin{aligned}
\|a - b\|^2 &= \left\| \sum_{i=1}^{k+1} \alpha_i x_i - \sum_{i=1}^{k+1} \beta_i x_i \right\|^2 \\
&= \left\| \sum_{i=1}^{k+1} \alpha_i x_i - x_{k+1} + x_{k+1} - \sum_{i=1}^{k+1} \beta_i x_i \right\|^2 \\
&= \left\| \sum_{i=1}^{k} \alpha_i (x_i - x_{k+1}) - \sum_{i=1}^{k} \beta_i (x_i - x_{k+1}) \right\|^2 \\
&= \left\| \sum_{i=1}^{k} (\alpha_i - \beta_i)(x_i - x_{k+1}) \right\|^2 \\
&= \sum_{i=1}^{k} (\alpha_i - \beta_i)^2 \|x_i - x_{k+1}\|^2.
\end{aligned}$$

similar,

$$\|f(a) - f(b)\|^2 = \sum_{i=1}^{k} (\alpha_i - \beta_i)^2 \|y_i - y_{k+1}\|^2.$$

So it follows from $\|x_i - x_{k+1}\| = \|y_i - y_{k+1}\|$, for all i, that $\|f(a) - f(b)\| = \|a - b\|$. Hence f is a convex isometrism. $\square$