The limit inferior and the limit superior

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October 29, 2021

Abstract

In this document I review some properties of the limit inferior and the limit superior: additivity properties, its relation the the limit and inequalities.

1 The limit inferior and limit superior

Definition 1. Let $(a_n)_{n\geq 0}$ be a sequence of real numbers. The limit inferior of a_n is $\liminf_{n\to\infty} a_n = \lim_{N\to\infty} \inf_{n\geq N} a_n$ and the limit superior of $(a_n)_{n\geq 0}$ is $\limsup_{n\to\infty} a_n = \lim_{N\to\infty} \sup_{n\geq N} a_n$.

Remark 2. As $((\inf_{n\geq N} a_n)_{N\geq 1}$ is an increasing sequence, and $((\sup_{n\geq N} a_n)_{N\geq 1}$ is a decreasing sequence, by the monotone convergence theorem, the limits $\lim_{N\to\infty} \inf_{n\geq N} a_n$ and $\lim_{N\to\infty} \sup_{n\geq N} a_n$ exist, but might be $\pm\infty$. Moreover, as $\inf_{n\geq N} a_n \leq \sup_{n\geq N} a_n$ for all $N \in \mathbb{N}$, it follows that

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

Remark 3. As $\sup_{n\geq N} a_n = -\inf_{n\geq N}(-a_n)$, we have that $\limsup_{n\to\infty} = -\liminf_{n\to\infty}(-a_n)$. This is an often convenient tool to translate lemma's about the limit inferior into lemma's about the limit superior, or vice versa.

2 Properties of the limit inferior

Lemma 4. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ sequences of real numbers. If $\liminf_{n\to\infty}(a_n - b_n) \geq 0$ then $\liminf_{n\to\infty} a_n \geq \liminf_{n\to\infty} b_n$.*

Proof. Suppose $\liminf_{n\to\infty}(a_n - b_n) \geq 0$, but for some $\varepsilon > 0$, $\liminf_{n\to\infty} a_n < -\varepsilon + \liminf_{n\to\infty} b_n$. Let $\delta > 0$. There is an N_0 so that for $N \geq N_0$,

$$\inf_{n\geq N} a_n < -\varepsilon + \delta + \inf_{n\geq N} b_n.$$

So for all $k \geq N$,

$$\inf_{n\geq N} a_n < -\varepsilon + \delta + b_k$$

So there is an $n \geq N$, so that for all $k \geq N$,

$$a_n < -\varepsilon + 2\delta + b_k.$$

In particular

$$a_n - b_n < -\varepsilon + 2\delta.$$

So

$$\inf_{k\geq N}(a_k - b_k) \leq a_n - b_n < -\varepsilon + 2\delta.$$

Taking $\delta \downarrow 0$, gives

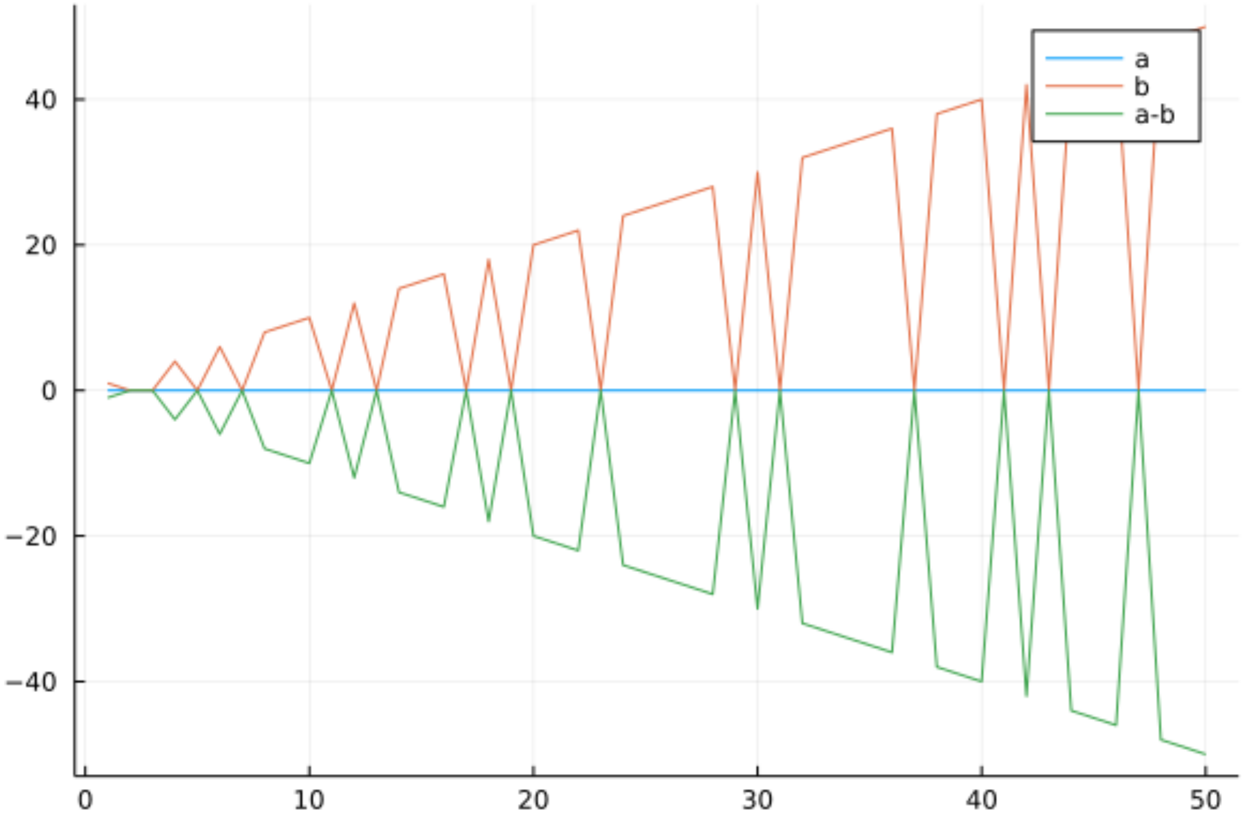
$$\liminf_{n\to\infty}(a_n - b_n) \leq -\varepsilon.$$

Contradiction. So $\liminf_{n\to\infty} a_n \geq -\varepsilon + \liminf_{n\to\infty} b_n$. As this holds for every $\varepsilon > 0$, we have that $\liminf_{n\to\infty} a_n \geq \liminf_{n\to\infty} b_n$. $\square$

The reverse is not true.

Lemma 5. *There are sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ of real numbers so that $\liminf_{n\to\infty} a_n \geq \liminf_{n\to\infty} b_n$, but $\liminf_{n\to\infty}(a_n - b_n) \not\geq 0$.*

Example 6. *Consider $(a_n)_{n\geq 0}$ with $a_n = 0$ for all n. Define $b_n = n$ except when n is prime, in which case $b_n = 0$.*



Then $\liminf_{n\to\infty} a_n = 0$, $\liminf_{n\to\infty} b_n = 0$, but $a_n - b_n$ is equal to $-n$ except, when n is prime, in which case it is equal to 0. Thus $\liminf_{n\to\infty}(a_n - b_n) = -\infty$, but $\liminf_{n\to\infty} a_n = 0 \geq 0 = \liminf_{n\to\infty} b_n$.

The limit inferior is also not additive.

Lemma 7. *There are sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ of (positive) real numbers so that $\liminf_{n\to\infty}(a_n + b_n) \neq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$.*

Example 8. *Define* $(a_n)_{n=1}^{\infty}$ *to be a sequence so that* $a_n = 1$ *when* n *is even and* $a_n = 2$ *when* n *is odd. Define* $(b_n)_{n=1}^{\infty}$ *to be a sequence so that* $b_n = 2$ *when* n *is even and* $b_n = 1$ *when* n *is odd. Then* $a_n + b_n = 3$ *for all* n*. So* $\liminf_{n\to\infty} a_n = \liminf_n b_n = 1$*, and* $\liminf_{n\to\infty}(a_n + b_n) = 3$*, so*

$$\liminf_{n\to\infty}(a_n + b_n) = 3 \neq 2 = \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.$$

However lim inf is superadditive:

Lemma 9. *Let* $(a_n)_{n\geq 1}, (b_n)_{n=1}^{\infty}$ *be sequences of real numbers. Then*

$$\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.$$

Proof. Let $N \in \mathbb{N}$. For all $k \geq N$,

$$a_k \geq \inf_{n\geq N} a_n \quad \text{and} \quad b_k \geq \inf_{n\to\infty} b_n.$$

So for all $k \geq N$,

$$a_k + b_k \geq \inf_{n\geq N} a_n + \inf_{n\geq N} b_n.$$

So

$$\inf_{n\geq N}(a_n + b_n) \geq \inf_{n\geq N} a_n + \inf_{n\geq N} b_n.$$

So

$$\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.$$

$\square$

Note that lemma 4 is now a corollary of lemma 9.

Second proof of lemma 4. It follows from lemma 9 that

$$\begin{aligned}\liminf_{n\to\infty} a_n &= \liminf_{n\to\infty}(b_n + (a_n - b_n)) \\ &\geq \liminf_{n\to\infty} b_n + \liminf_{n\to\infty}(a_n - b_n) \\ &\geq \liminf_{n\to\infty} b_n,\end{aligned}$$

by our assumption that $\liminf_{n\to\infty}(a_n - b_n) \geq 0$. $\square$

Lemma 10. *Let* $(a_n)_{n=1}^{\infty}$ *be a sequence of real numbers and let* $(b_n)_{n=1}^{\infty}$ *be a sequence of real numbers that converges to* $b \in \mathbb{R}$*. Then*

$$\liminf_{n\to\infty}(a_n + b_n) = b + \liminf_{n\to\infty} a_n.$$

Proof. First consider the case that $\liminf_{n\to\infty} a_n = \infty$ or $-\infty$, in which case $\inf_{n\geq N} a_n$ converges to ∞ or $-\infty$ as well, respectively. Note that $(b_n)_{n=1}^{\infty}$ is bounded, say $|b_n| \leq B$ for all n. So for all $N \in \mathbb{N}$, $-B + \inf_{n\geq N} a_n \leq \inf_{n\geq N}(a_n + b_n) \leq B + \inf_{n\geq N} a_n$, hence $\inf_{n\geq N}(a_n + b_n)$ converges to ∞ or $-\infty$ as well.

Now consider the case that $a := \liminf_{n\to\infty} a_n \in \mathbb{R}$. Let $\varepsilon > 0$. Then there is an $N_0 \in \mathbb{N}$, so that for $N \geq N_0$ and $n \geq N$, $|\inf_{n\geq N} a_n - a| < \varepsilon/4$ and $|b_n - b| < \varepsilon/2$. So for all $N \geq N_0$ and for all $k \geq N$, we have

$$a - \varepsilon/4 < \inf_{n\geq N} a_n \leq a_k$$

and there is a $M_{N,\varepsilon} \geq N$, so that

$$\left| \inf_{n \geq N} a_n - a_{M_{N,\varepsilon}} \right| < \varepsilon/4.$$

So

$$\left| a_{M_{N,\varepsilon}} - a \right| < \varepsilon/2.$$

It follows that for all $k \geq N$,

$$a + b - \frac{3}{4}\varepsilon < a_k + b_k,$$

and

$$a_{M_{N,\varepsilon}} + b_{M_{N,\varepsilon}} \leq a + b + \varepsilon.$$

It follows that for all $N \geq N_0$,

$$a + b - \frac{3}{4}\varepsilon \leq \inf_{n \geq N} (a_n + b_n) \leq a + b + \varepsilon.$$

Taking the limit $\varepsilon \downarrow 0$ gives

$$\liminf_{n \to \infty} (a_n + b_n) = a + b.$$

□

2.1 Statements about the limit superior

We will now translate our statements for the limit inferior to statements about the limit superior, with the use of remark 3.

The equivalent to lemma 4 is

Lemma 11. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of real numbers. If $\limsup_{n \to \infty} (a_n - b_n) \leq 0$, then $\limsup_{n \to \infty} a_n \leq \limsup_{n \to \infty} b_n$.*

Proof. Note that

$$\limsup_{n \to \infty} (a_n - b_n) = -\liminf_{n \to \infty} (b_n - a_n) \leq 0,$$

So $\liminf_{n \to \infty} (b_n - a_n) = \liminf_{n \to \infty} ((-a_n) - (-b_n)) \geq 0$. Hence, according to lemma 4,

$$\liminf_{n \to \infty} (-a_n) \geq \liminf_{n \to \infty} (-b_n).$$

So

$$-\liminf_{n \to \infty} (-a_n) \leq -\liminf_{n \to \infty} (-b_n).$$

So

$$\limsup_{n \to \infty} a_n \leq \limsup_{n \to \infty} b_n.$$

□

The equivalent to lemma 5 is

Lemma 12. *There are sequences* $(a_n)_{n\geq 0}$*,* $(b_n)_{n\geq 0}$ *of real numbers so that* $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$*, but* $\limsup_{n\to\infty}(a_n - b_n) \not\leq 0$*.*

Proof. According to example 6 there are sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ so that $\liminf_{n\to\infty} a_n \geq \liminf_{n\to\infty} b_n$, but $\liminf_{n\to\infty}(a_n - b_n) < 0$. So

$$-\limsup_{n\to\infty}(-a_n) \geq -\limsup_{n\to\infty}(-b_n),$$

that is

$$\limsup_{n\to\infty}(-a_n) \leq \limsup_{n\to\infty}(-b_n)$$

but

$$-\limsup_{n\to\infty}((-a_n) - (-b_n)) < 0,$$

so

$$\limsup_{n\to\infty}((-a_n) - (-b_n)) > 0.$$

So $(-a_n)_{n=1}^{\infty}$ and $(-b_n)_{n=1}^{\infty}$ are the required sequences. □

Similar to the limit inferior (lemma 7), the limit superior is also not additive.

Lemma 13. *There are sequences* $(a_n)_{n\geq 0}$ *and* $(b_n)_{n\geq 0}$ *of (positive) real numbers so that* $\limsup_{n\to\infty}(a_n + b_n) \neq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$*.*

This is proven by the following counterexample, which has the same sequences as example 8

Example 14. *Let* $(a_n)_{n=1}^{\infty}$ *and* $(b_n)_{n=1}^{\infty}$ *be as in lemma 7, so* $a_n = 1$ *when* n *is even and* $a_n = 2$ *when* n *is odd and* $b_n = 2$ *when* n *is even and* $b_n = 1$ *when* n *is odd. Then* $a_n + b_n = 3$ *for all* n*. So* $\limsup_{n\to\infty} a_n = \limsup_n b_n = 2$*, and* $\limsup_{n\to\infty}(a_n + b_n) = 3$*, so*

$$\limsup_{n\to\infty}(a_n + b_n) = 3 \neq 4 = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

As the limit inferior is superadditive (lemma 9), the limit superior is subadditive:

Lemma 15. *Let* $(a_n)_{n=1}^{\infty}$*,* $(b_n)_{n=1}^{\infty}$ *be sequences of real numbers. Then*

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Proof. Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ be sequences of real numbers. By lemma 9,

$$\begin{aligned}\limsup_{n\to\infty}(a_n + b_n) &= -\liminf_{n\to\infty}(-a_n + -b_n)\\ &\leq -\liminf_{n\to\infty}(-a_n) + -\liminf_{n\to\infty}(-b_n)\\ &= \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.\end{aligned}$$

□

The equivalent of lemma 10 is

Lemma 16. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers and let $(b_n)_{n=1}^{\infty}$ be a sequence of real numbers that converges to $b \in \mathbb{R}$. Then*

$$\limsup_{n\to\infty}(a_n + b_n) = b + \limsup_{n\to\infty} a_n.$$

Proof. Note that $(-b_n)_{n=1}^{\infty}$ converges to $-b$, so by lemma 10,

$$\begin{aligned}\limsup_{n\to\infty}(a_n + b_n) &= -\liminf_{n\to\infty}(-a_n - b_n)\\ &= -\left(-b + \liminf_{n\to\infty}(-a_n)\right)\\ &= b + \limsup_{n\to\infty} a_n.\end{aligned}$$

□

Lemma 17. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Then $(a_n)_{n=1}^{\infty}$ converges in $[-\infty, \infty]$ if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$. Moreover, when $(a_n)_{n=1}^{\infty}$ converges, then*

$$\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n \in [-\infty, \infty].$$

Proof. First suppose $(a_n)_{n=1}^{\infty}$ converges to ∞. Then for every $M > 0$ there is an $N_0 \in \mathbb{N}$ so that for all $N \geq N_0$ and for all $n \geq N$, $a_n \geq M$. So $\sup_{n\geq N} a_n \geq M$ and $\inf_{n\geq N} a_n \geq M$. Taking $M \uparrow \infty$ gives

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \infty.$$

Now suppose that $(a_n)_{n=1}^{\infty}$ converges to $-\infty$. Then for every $M > 0$ there is an $N_0 \in \mathbb{N}$ so that for $N \geq N_0$ and for all $n \geq N$, $a_n \leq -M$. So $\inf_{n\geq N} a_n \leq -M$ and $\sup_{n\geq N} a_n \leq -M$. Taking the limit $M \uparrow \infty$ gives

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = -\infty.$$

Now suppose $(a_n)_{n=1}^{\infty}$ converges to $a \in \mathbb{R}$. Let $\varepsilon > 0$. Then there is an $N_0 \in \mathbb{N}$ so that for $N \geq N_0$ and for $n \geq N$, $|a_n - a| < \varepsilon$. Thus $\inf_{n\geq N} a_n \geq a - \varepsilon$ and $\sup_{n\geq N} a_n \leq a + \varepsilon$. Taking the limit $\varepsilon \downarrow 0$ gives

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = a.$$

Conversely, suppose $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \infty$. In particular $\inf_{n\geq N} a_n$ diverges to infinity. So for every $M > 0$ there is some $N \in \mathbb{N}$ so that for $n \geq N$, $a_n \geq M$. So

$$\lim_{n\to\infty} a_n = \infty.$$

So

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \infty.$$

Now suppose $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = -\infty$. In particular $\sup_{n\geq N} a_n$ diverges to minus infinity. So for every $M > 0$ there is some $N \in \mathbb{N}$ so that for $n \geq N$, $a_n \leq -M$. So $\lim_{n\to\infty} a_n = -\infty$. So

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = -\infty.$$

Now suppose $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a \in \mathbb{R}$. Let $\varepsilon > 0$. Then there is an N_0 so that for all $N \geq N_0$,

$$a - \varepsilon \leq \inf_{n\geq N} a_n \leq \sup_{n\geq N} a_n \leq a + \varepsilon.$$

So for all $n \geq N_0$,

$$a - \varepsilon \leq a_n \leq a + \varepsilon.$$

Taking $\varepsilon \downarrow 0$ gives $\lim_{n\to\infty} a_n = a$. So

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = a.$$

$\square$